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Effects of degree distribution in mutual synchronization of neural networks

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We study the effects of the degree distribution in mutual synchronization of two-layer neural networks. We carry out three coupling strategies: large-large coupling, random coupling, and small-small coupling. By computer simulations and analytical methods, we find that couplings between nodes with large degree play an important role in the synchronization. For large-large coupling, less couplings are needed for inducing synchronization for both random and scale-free networks. For random coupling, cutting couplings between nodes with large degree is very efficient for preventing neural systems from synchronization, especially when subnetworks are scale-free.

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Synchronization phenomena in neural systems have attracted much attention. These phenomena are thought to be important for functioning neural system, such as neural coding, visual information processing, sleeping, and memory in brain [1, 2, 3, 4]. Besides the complete synchronization, which results in identical states of all neurons in a uniform population, more subtle forms of synchronization should be present for examining the brain functions. The brain is essentially a system of interacting neural networks and the activity pattern of different networks may become synchronized while retaining their complex spatiotemporal dynamics. Therefore, it is interesting to investigate mutual synchronization in ensembles of coupled neural networks [5]. This problem has been studied on fully connected networks [6] and on random networks [7].

Recently, it was suggested that connectivity in neural systems is more complex [8]. The effects of complicated topologies on network dynamics have been confirmed in some theoretical studies. For instance, small-world neural networks give rise to fast system response with coherent oscillations [9]. Scale-free Hopfield networks can recognize blurred pattern efficiently [10]. Complex networks have both sensitivity and robustness as responding to different stimuli [11]. On the other hand, synchronization is not always desired in neural systems. For instance, several neurological diseases such as Parkinson's disease and epilepsy are caused by synchronized firing of neural oscillators [12]. So it is also important to study desynchronization and instability of synchronized motion of neural systems.

In this paper, we investigate the efficiency of scale-free topology in inducing synchronization and preventing the system from synchronization in two-layer neural networks. We carry out three coupling strategies between subnetworks: large-large coupling (couplings built between nodes with large degree); random coupling (couplings built between randomly selected nodes); small-small coupling (couplings built between nodes with small degree). Computer simulations reveal that couplings between nodes with large degree play an important role either in inducing synchronization or preventing the system from synchronization. An analytical treatment confirms the numerical simulation result.

We consider a neural network model that consists of N neurons $x_i(t) \in (-1,1)$, $i = 1, \cdots, N$. The topology of networks was represented by symmetric adjacency matrix A whose entry a_{ij} $(i,j = 1,\cdots,N)$ is equal to 1 when neuron i connects to neuron j, and zero otherwise. Each link has a weight J_{ij} which is a random number distributed uniformly in the interval between -1 and 1. The system considered is composed of two identical neural networks, and a part of corresponding nodes in different subsystems is coupled together. The dynamics of the system is described by the following equations [6, 7],

$$\begin{aligned} x_i^1(t+1) &= (1-\varepsilon_i)\Theta(h_i^1(t)) + \varepsilon_i\Theta(h_i^1(t)+h_i^2(t)), \\ x_i^2(t+1) &= (1-\varepsilon_i)\Theta(h_i^2(t)) + \varepsilon_i\Theta(h_i^1(t)+h_i^2(t)). \end{aligned} \quad (1)$$

In the equation, ε_i represents the coupling strength between the nodes i in different networks. When a pair of nodes is coupled, the coupling strength between them is equal to a constant $\varepsilon_i = \varepsilon$, otherwise $\varepsilon_i = 0$. For large-large (small-small) coupling we choose a group of nodes which consists of the nodes of greatest (smallest) degree in a subnetwork, and take $\varepsilon_i = \varepsilon$ if node i is in the group. Here $h_i^l(t)$ is the local field of the ith neuron and is expressed by

$$h_i^l = \sum_{j=1}^{N} a_{ij}J_{ij}x_j^l(t),\ l = 1,2 \quad (2)$$

denoting the signal arriving at neuron i at time t from neurons of the same network. The local field is determined by the network topology. The non-coupled nodes in each subnetwork interact indirectly with another subnetworks through the local field. The activity function $\Theta(r)$ is defined by

$$\Theta(r) = [1+\tanh(\beta r)]/2, \quad (3)$$

where $\beta = 1/T$ characterizes a measure of the inverse magnitude of the amount of noise affecting this neuron, performing the role of reciprocal temperature in analogy to thermodynamic systems. For convenience, we set $\beta = 10$ through

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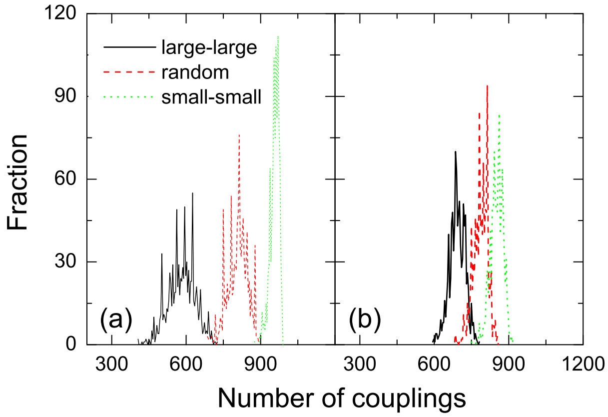

FIG. 1: (color online) Histograms of the number of couplings needed by the system to reach synchronization for three coupling strategies: large-large coupling (solid), random coupling (dash), and small-small coupling (dot), respectively. The system is composed of BA networks (a) or ER networks (b) with $N = 1000$, $\langle k \rangle = 20$, and $\varepsilon = 0.6$. 1000 simulations were performed for each histogram.

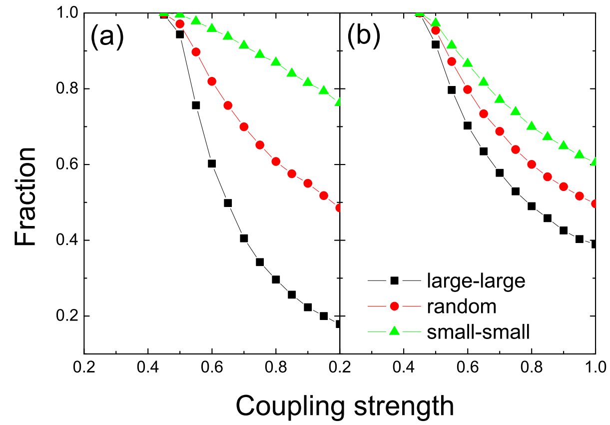

FIG. 2: (color online) The number of couplings (fractions of total nodes) needed for reaching synchronization as a function of the coupling strength ε for three coupling methods. The system is made up of BA networks (a) or ER networks (b). Parameters: $N = 1000$, $\langle k \rangle = 20$.

all simulations. The initial states of all neurons in two subsystems are randomly chosen. To measure coherence in this coupled system, the dispersion of activity patterns is introduced

$$D(t) = \frac{1}{2} \sum_{l=1}^{2} \sum_{i=1}^{N} [x_i^l(t) - \overline{x}_i(t)]^2, \tag{4}$$

where $\overline{x}_i(t) = \sum_{l=1}^{2} x_i^l(t)/2$ is the average activity of neurons occupying the position i in both subnetworks at time t. The dispersion vanishes when the system reaches the completely synchronous state.

Firstly, we investigate the number of couplings needed between two subnetworks for system synchronization to study the efficiency of the network topology. In Fig. 1, we plot the histograms of the number of couplings built to guarantee synchronization for scale-free and random networks. We use Barabási-Albert (BA) arithmetic [13] to generate the scale-free network and use Erdös-Rényi (ER) arithmetic [14] to construct the random network. According to the parameter setting in [11], we also choose the size ($N = 1000$) and average degree ($\langle k \rangle = 20$) for both kinds of networks. When subnetworks are scale-free (see Fig. 1(a)), the mean fraction of couplings are 58.2%, 80.7%, and 95.6%, corresponding to the large-large coupling, random coupling, and small-small coupling, respectively. This implies that the large-large coupling strategy is more efficient than the random coupling and the small-small coupling is the most inefficient method for inducing synchronization. In addition, the small-small coupling can be regarded as removing couplings which link nodes with larger degree from the globally coupled system. Therefore, removing couplings among nodes with larger degree at the initial state can efficiently prevent the system from synchronizing. In contrast to the case of scale-free topology, the peaks corresponding to three coupling methods for the system consisted of random networks are more closer (see Fig. 1(b)), which is caused by the homogeneous distribution of network connectivity. This implies that the topology of subnetworks and the degree of coupled nodes influence the efficiency of the system, and the scale-free topology is more efficient than random network for inducing synchronization or preventing the system from synchronized states.

Figure 2 shows the fraction of couplings needed to induce synchronization versus the coupling strength ε. Whether subnetworks are scale-free or random, there is a critical point $\varepsilon_c = 0.44$ below which partial coupled networks can not synchronize. For $\varepsilon > \varepsilon_c$, degrees of nodes taking part in interactions between two subnetworks will efficiently affect synchronization or the prevention of networks from synchronization. The larger the parameter ε is, the less couplings are needed for system synchronization. Furthermore, given the coupling strength, the scale-free topology is more efficient than the random graph to synchronize in the case of large-large coupling.

We now examine the dependence of the fraction of couplings needed for synchronization on the average links per nodes $\langle k \rangle$ in a subnetwork. For small $\langle k \rangle$, due to few neighbors per node and weak indirect interaction between non-coupled nodes, more couplings are needed for networks to synchronize. For large $\langle k \rangle$, different from classical results of network synchronization [15], the coupled system still needs many nodes to be coupled to ensure synchronization. Figure 3 displays a surprisingly nonmonotonic dependence of the fraction on $\langle k \rangle$ for given network size and the coupling strength, i.e., there exists an optimal value of $\langle k \rangle$ for which the fraction of couplings reaches minimum. With the increase of $\langle k \rangle$, the interaction of nodes inside a subnetwork is enhanced through the local field. The enhanced inside interaction brings the big indirect interaction between non-coupled nodes, which helps

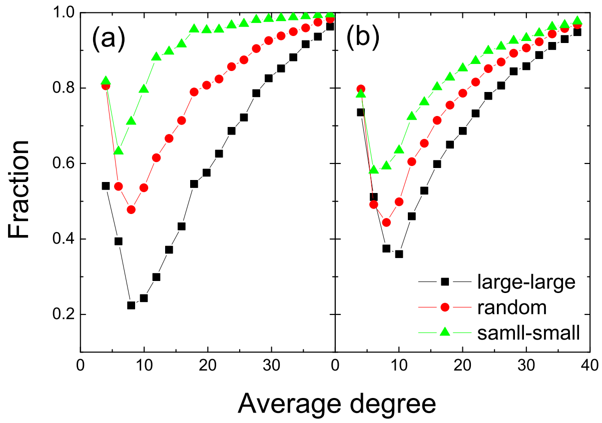

FIG. 3: (color online) The dependence of the fraction of couplings needed for reaching synchronization on the average degree of subnetworks, BA networks (a) or ER networks (b). Parameters: $N = 1000, \varepsilon = 0.6$.

to the synchronization of non-coupled nodes. So less couplings are needed for system synchronization when $\langle k \rangle$ increases. On the other hand, the evolution of subsystems with stronger inside interaction is more stable and it needs more couplings to drive their evolutions into synchronization [7]. The indirect interaction between non-coupled nodes and the stability of subnetworks get a proper match, when $\langle k \rangle$ takes the optimal value.

Next, we will demonstrate the difference between the efficiency of scale-free and random networks for inducing or preventing synchronization. To determine the fraction of couplings needed for synchronization, we consider a node which does not take part in direct interaction between two subnetworks. The state of this node is determined by its local field, $h_i(t)$, defined by Eq. (2). The local field includes two types of signals, one comes from neighbors which interact with nodes of another subnetwork and the other comes from the rest neighbors. Thus the total degree of the coupled nodes determines whether the system can reach the synchronized state through the intensity of signals in the mean local field. For the the six schemes investigated in Fig. 1, we calculated the critical values k_c of the total degree of the coupled nodes when synchronization occurs. These k_c are normalized by the sum of the degree of the subnetwork $N * \langle k \rangle$. The standard deviation of the critical values is 0.019, while the mean of critical total degree of coupled nodes is 0.785. So we argue that synchronization will occur if the sum of the degrees of coupled nodes in one subnetwork exceeds a threshold.

For convenience of notation, the number of coupled nodes is labelled to be F_r for random coupling and F_l for large-large coupling. When the system is randomly coupled by either scale-free or random subnetworks, the average degree of coupled nodes is nearly the same as the average degree of the subnetworks $\langle k \rangle$. So the total degree of coupled nodes is

$$\sum_{i=1}^{F_r} k_i = \langle k \rangle F_r, \tag{5}$$

where k_i is the degree of neuron i. For any network topology with the same average degree and size, the fraction of couplings is the same in the case of random coupling. Following ideas developed by Bar-Yam and Epstein [11], we get the relation between the fraction of couplings for random coupling (denoted by f_r, $f_r = F_r/N$) and the fraction of couplings for large-large coupling (denoted by f_l, $f_l = F_l/N$).

When subnetworks are random, the total degree of coupled nodes is

$$\sum_{i=1}^{F_l} k_i = N \sum_{k_l}^{\infty} \frac{k \langle k \rangle^k e^{-\langle k \rangle}}{k!} \tag{6}$$

for large-large coupling, where k_l denotes the minimum degree of the coupled nodes. Thus the fraction of couplings is

$$f_l = \sum_{k_l}^{\infty} \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!} \tag{7}$$

So the difference of the fraction of couplings for two coupled methods is

$$f_r - f_l = \frac{\langle k \rangle^{k_l - 1} e^{-\langle k \rangle}}{(k_l - 1)!}, \tag{8}$$

which maximum over k_l is obtained approximately by setting $k_l = \langle k \rangle + 1/2$ for a given value $\langle k \rangle$ [11].

When subnetworks are scale-free, the degree distribution has a power law shape $P(k) = Ak^{-\gamma}$. The total degree of coupled nodes is

$$\begin{aligned} \sum_{i=1}^{F_l} k_i &= N \int_{k_l}^{\infty} kP(k)\mathrm{d}k \\ &= \frac{1}{\gamma - 2} N A k_l^{2-\gamma} \end{aligned} \tag{9}$$

with the ancillary condition

$$F_l = N \int_{k_l}^{\infty} P(k)\mathrm{d}k = \frac{1}{\gamma - 1} N A k_l^{1-\gamma} \tag{10}$$

for large-large coupling. Normalizing the probability distribution and assuming a sharp cutoff of the distribution at low k, we yield

$$A = \frac{(\gamma - 2)^{(\gamma-1)}}{(\gamma - 1)^{(\gamma-2)}} \langle k \rangle^{(\gamma-1)}. \tag{11}$$

Combining Eqs. (6), (9) and (11), the relation between f_l and f_r is obtained

$$f_l = f_r^{(\gamma-1)/(\gamma-2)}. \tag{12}$$

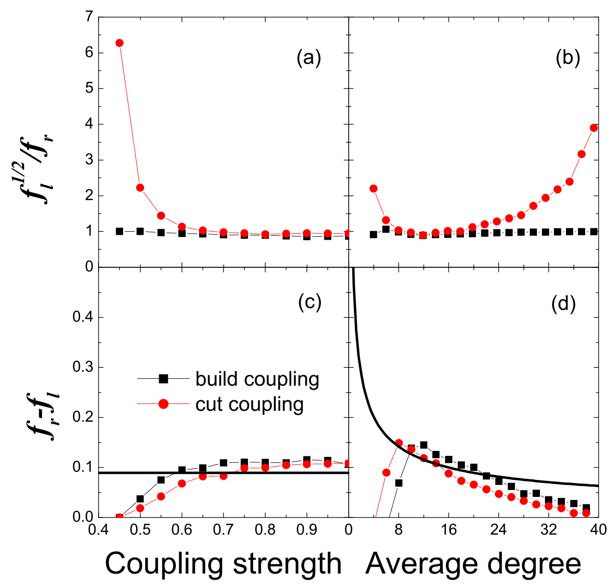

FIG. 4: (color online) Comparison numerical results with analytical predictions for building couplings to induce synchronization (square) or cutting couplings to prevent the system from synchronization (circle). Upper panel: the ratio $f_l^{1/2}/f_r$ for scale-free subnetworks as of function of the coupling strength (a) and of the average degree of subsystems (b). Lower panel: the difference $f_r - f_l$ for random subnetworks as a function of the coupling strength (c) and of the average degree of subsystems (d). The thick lines are the maximum of $f_r - f_l$ obtained by analytical calculations.

For BA model, the degree exponent γ is equal to 3 [13]. Thus we have $f_l = f_r^2$.

There also exists a threshold of degree for preventing the system from synchronization when we cut couplings from the globally coupled system. This threshold is equal to the difference between $N\langle k\rangle$ and the threshold of degree for ensuring synchronization. Similarly, the relationship between the fraction of removed couplings for large-large cutting and random cutting also follow Eqs. (8) and (12).

Figure 4(a) shows $f_l^{1/2}/f_r$ as a function of coupling strength when subnetworks are scale-free. In the case of building couplings, numerical simulations give that $f_l^{1/2}/f_r = 1$, which is consistent with Eq. (12). In the case of cutting couplings, the ratio is larger than the analytical prediction when the coupling strength ε is small, which results from the fat tail [16] of BA networks. In other words, the number of nodes with large degree is more than that described by the power-law distribution. As a result, some extra couplings between large degree nodes are removed in simulations and therefore the number of cut couplings predicted by analysis is less than that of simulations. When coupling strength is strong, the number of couplings cut from the coupled system is large, and the simulation results close to the analytical prediction. Figure 4(b) shows the ratio $f_l^{1/2}/f_r$ as a function of the average degree $\langle k\rangle$ of scale-free subsystems. For building couplings, simulation results agree well with the analytical prediction. For cutting couplings, the simulation results are greater than the analytical calculation in the case of either $\langle k\rangle$ is low or large. The deviation results from the small fraction of couplings in these regions as shown in Fig. 3(a). When subnetworks are random, the difference $f_r - f_l$ are shown in Figs. 4(c) and 4(d) as a function of the coupling strength ε and the average degree of subnetworks $\langle k\rangle$. The analytical result of the upper boundary of $f_r - f_l$ gives a good limitation to numerical results. Although building large-large couplings and removing large couplings improve the efficiency in inducing and preventing synchronization, the analytical result of random subnetworks restricts the enhancement of efficiency to a small range which is less than that of scale-free subnetworks.

In summary, we have studied the influences of the degree distribution of networks on mutual synchronization in a two-layer neural networks. We investigated three coupling methods between two subsystems: large-large coupling, random coupling, and small-small coupling. We found that couplings between nodes with large degree nodes play an important role in the synchronization. For large-large coupling, less couplings are needed for inducing synchronization for both random and scale-free networks. For random coupling, cutting couplings between nodes with large degree is very efficient for preventing neural systems from synchronization, especially when subnetworks are scale-free. By assuming that the total degree of coupled nodes in subnetworks determines the system synchronization, the numerical simulation results are interpreted analytically. The analysis reveals that the degree distribution of subnetworks rather than other topological quantities affects the efficiency of systems in synchronization. Although our work is based on a simple model of neural systems, we think that the results found out in this work is proper in more wide and realistic situations in which the dynamics of neurons depend on the mean local field. It would be interesting if Nature takes advantage of the efficiency of the scale-free topology in controlling mutual synchronization of interacted systems.

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